An approximate analytical solution description of time-fractional order Fokker-Planck equation by using FRDTM

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ABSTRACT

The fractional Fokker-Plank equation paid much more attention by researchers due to its wide range of applications in several areas of sciences and engineering. In this article, our main motto is to present a new approximate solution of time-fractional FPE by means of a new semi-analytical approach called fractional reduced differential transform method (FRDTM), with appropriate initial condition. In FRDTM, fractional derivative is considered in the Caputo sense. The validity and efficiency of FRDTM is illustrated by considering three numeric experiments. The solutions behavior and the effects of different values of fractional order are depicted graphically.

Keywords: Fokker-Planck equation; FRDTM; Caputo time derivative; exact solution

1. Introduction

In the recent years, fractional calculus theory gained a great attention in both sciences and engineering [1-10] due to its application to model many real world problems and fractional dynamical systems, e.g., in earthquake modeling, measurement of viscoelastic material properties, the traffic flow model, fluid flow model with fractional derivatives etc. The fractional order system equation being converges to the integer order equation, have also paid more attention to the researchers. Risken [11] have first time described the Brownian motion of particles by mean of Fokker-Planck equations (FPE) as one of dynamical equation. The simplest FPE equation for the motion of a small particle of mass $m$ immersed in a fluid of temperature $T$ is given as

\begin{equation}
p \frac{\partial \rho}{\partial t} + \nabla \cdot (D \nabla \rho) = \beta \rho, \quad \rho(x,0) = \rho_0(x)
\end{equation}
\[ D_t^1 u = \gamma D_t^1 (vu) + \gamma (KT/m) D_t^2 u, \]

where \( D_t^1 u = \frac{\partial u}{\partial t} \), \( D_t^2 u = \frac{\partial^2 u}{\partial v^2} \); and \( u(x,t) \) and \( v \) are the distribution function and the velocity for the Brownian motion of the particle, respectively, \( K \) denotes the Boltzmann’s constant and \( \gamma \) the fraction constant. The forward Kolmogorov equation, a more general form of FPE (1), representing the motion for the distribution function \( u(x,t) \), can be expressed as

\[ D_t^1 u = \left[-D_t^1 A(x) + D_t^2 B(x)\right] u, \quad \forall \ x \in I, t > 0 \]

subject to the initial condition \( u(x,0) = \phi(x), \forall \ x \in I \), where \( I = (a,b) \) and \( \phi \) is known function and \( A(x) \) denotes the drift coefficient while \( B(x) > 0 \) the diffusion coefficient. These coefficients may be function of time \( t \), that is,

\[ D_t^1 u = \left[-D_t^1 A(x,t) + D_t^2 B(x,t)\right] u. \]  

(3)

It is clear that Eq. (1) is a special case of (3), when \( A(x,t) \) is linear in \( x \) and \( B(x,t) \) is constant. Risken [11] has introduced a similar partial differential equation, a more general form of FPE, called Backward Kolmogorov equation is given by

\[ D_t^1 u = \left[-A(x,t) D_t^1 + B(x,t) D_t^2\right] u, \]

(4)

The nonlinear time-fractional Fokker-Planck equations of order \( \alpha \) \((0 < \alpha \leq 1)\) is expressed as

\[ D_t^\alpha u = \left[-D_t^1 A(x,t,u) + D_t^2 B(x,t,u)\right] u. \]

(5)

Eq. (5) is reduced to the most general form classical nonlinear FPE when \( \alpha = 1 \). The FPE have its wide range of applications in several areas such as solid-state physics, theoretical biology, circuit theory, quantum theory, chemical physics, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, engineering, pattern formation, psychology and marketing and so on (see [12] and the references therein). The most important role of FPE is in describing the system dynamics [13–26], e.g. the dynamics of energy cascade in turbulence [13], the population dynamics [14], the fission dynamics [15] the chaotic universe dynamics [16], the fatigue crack growth dynamics [17], the Langevin approach for microscopic dynamics [18], the dynamics of distributions of heavy quarks [19], financial returns [20], the electron dynamics of plasmas and semiconductors [21], the spin relaxation dynamics [22], the critical dynamics [23], and the fiber dynamics [24] had been investigated by using the Fokker-Planck equation, see [25, 26] and the references therein. The fractional diffusion equations got their attention among the researchers due to its arising in the modeling of anomalous diffusive and sub-diffusive systems, continuous time random walks, unification of diffusion and wave propagation phenomenon and so on [27]. The fractional FPE have been intensively studied by several researchers in [28–42] due to their broad applications.
The numeric and analytical solutions of FPE have been intensively studied since the work of Risken [11], and the various approaches have been developed for getting the analytic [43-47] and numeric [47-50] solution of FPE. Some of these schemes are moving finite element method [43], differential transform method (DTM) [44], variational iteration method (VIM) and homotopy-perturbation methods (HPM) [45]. Adomian decomposition method (ADM) [46], cubic B-spline method [47] and so on.

The fractional order FPE has been solved analytically by means of iterative Laplace Transform method [27], homotopy perturbation method (HPM) [28], finite element method [39] and homotopy perturbation transform method (HPTM) [41], for more schemes, see [28-42] and references therein. The major drawback of these approaches is that they require a very complicated and huge calculation. The fractional reduced differential transform method (FRDTM) is proposed by Keskin and Oturanc [51] to overcome from such type of drawbacks. This FRDTM is very easy to implement as it takes small size of computation contrary to the other numerical methods, in the literature. The method is very reliable, effective and efficient powerful approach applicable to a wide range of problems arising in sciences and engineering, see [52-54]. In this paper, our main motto is to develop a new approximate analytical solutions approach for nonlinear time-fractional Fokker-Planck equations of order $\alpha$ $(0 < \alpha \leq 1)$ by means of FRDTM, in the series form converges to the exact solution rapidly.

The rest of the paper is organized as follows: in Section 2, we revisit some basic preliminaries and notations based on fractional calculus theory while Section 3 is the basics of FRDTM which are used for further study. In Section 4, three test problems of time-fractional homogenous and non-homogeneous fractional Fokker Planck equations are presented. The comparison is made with the exact solutions available, in the literature. The concluding remarks are presented in Section 5.

2. Fractional Calculus Theory

The basic preliminaries are revisited in this section which we use in this paper. Among several results based on fractional integrals and derivatives proposed, in the literature, we consider the first major contribution to give a proper and most meaningful result proposed by Liouville [3].

**Definition 2.1** A real valued function $f(x) \in \mathbb{R}, x > 0$ is said to be in the space $C_{\mu}, \mu \in \mathbb{R}$ if there exists a real number $q \ (> \mu)$ such that $f(x) = x^q g(x)$ , where $g(x) \in C[0,\infty)$ , and is said to be in the space $C_{\mu}^m$ if $f^{(m)} \in C_{\mu}, m \in \mathbb{N}$ .

**Definition 2.2** Let $f \in \mathbb{R}$ . Riemann-Liouville fractional integral operator $^\alpha$ of order $\alpha \geq 0$ is defined by
\[
J_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt, \quad \alpha > 0, x > 0,
J_0^0 f(x) = f(x).\tag{6}
\]

In his work, Caputo and Mainardi\cite{2} proposed a modified fractional differentiation operator \(D_x^\alpha\) on the theory of visco-elasticity to overcome the discrepancy of Riemann-Liouville derivative \cite{3} while modeling the real world problems using the fractional differential equations. They further, demonstrated that their proposed Caputo fractional derivative allow the utilization of initial and boundary conditions involving integer order derivatives, is a straightforward physical interpretation.

**Definition 2.3** The fractional derivative of \(f(x) \in \mathbb{R}\), in the Caputo sense \cite{2} is defined as

\[
D_x^\alpha f(x) = J_x^{m-\alpha} D_x^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) \, dt,
\]

for \(m-1 < \alpha \leq m\), \(m \in \mathbb{N}\), \(x > 0\), \(f \in C_m^m\). The basic properties of the Caputo fractional derivative can be given by the following

**Lemma 2.1** If \(m-1 < \alpha \leq m\), \(m \in \mathbb{N}\) and \(f \in C_m^m\), \(\mu \geq -1\), then we have

\[
\begin{align*}
D_x^\alpha J_x^\alpha f(x) &= f(x), \quad x > 0, \\
J_x^\alpha D_x^\alpha f(x) &= f(x) - \sum_{k=0}^{m} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0.
\end{align*}
\tag{8}
\]

In the present work, the Caputo fractional derivative is considered because it allows the traditional initial and boundary conditions to be included in the formulation of the physical problems, for further important characteristics of fractional derivatives, see \cite{1-10}.

### 3. The Basic Idea of FRDTM

In this section, the basic properties of the fractional reduced differential transform method are described. Let \(\psi(x,t)\) be a function of two variables, which can be represented as a product of two single-variable functions, that is \(\psi(x,t) = f(x) \cdot g(t)\). Using the properties of the one-dimensional differential transform (DT) method, \(\psi(x,t)\) can be written as

\[
\psi(x,t) = \sum_{i=0}^{\infty} f(i)x^i \sum_{j=0}^{\infty} g(j)t^j = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Psi(i, j)x^it^j,
\tag{9}
\]
where $\Psi(i, j) = f(i)g(j)$ is referred to as the spectrum of $\psi(x, t)$.

Let $R_D$ and $R_D^{-1}$ denotes operators for fractional reduced differential transform (FRDT) and inverse FRDT, respectively. The basic definition and properties of the FRDTM is described below.

Lemma 3.1 If $\psi(x, t)$ is analytic and continuously differentiable with respect to space variable $x$ and time variable $t$ in the domain of interest, then the $t$-dimensional spectrum is given by

$$
\Psi_k(x) = \frac{1}{\Gamma(k\alpha + 1)} \left[ D_t^k(\psi(x,t)) \right]_{t=t_0},
$$

is referred as the FRDT of $\psi$, where the parameter $\alpha$ describes the order of time-fractional derivative. Throughout the paper, lowercase $\psi(x, t)$ is used for the original function while its fractional reduced transformed function is represented by the uppercase $\Psi(x)$.

The inverse FRDT of $\Psi_k(x)$ is defined as

$$
\psi(x, t) := \sum_{k=0}^\infty \Psi_k(x)(t-t_0)^{k\alpha}.
$$

Using Eq. (10) and (11), the following results can be obtained

$$
\psi(x, t) = \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \left[ D_t^k(\psi(x,t)) \right]_{t=t_0} (t-t_0)^{k\alpha}.
$$

In particular, for $t_0 = 0$, Eq. (12) reduces to

$$
\psi(x, t) = \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \left[ D_t^k(\psi(x,t)) \right]_{t=0} t^{k\alpha}.
$$

This shows that FRDTM is a special case of the power series expansion of a function.

Let us suppose $u(x, t)$ and $v(x, t)$ be any functions such that $u(x, t) = R_D\left[ U_k(x) \right]$, $v(x, t) = R_D\left[ V_k(x) \right]$ and the convolution $\otimes$ denotes the fractional reduced differential transform version of the multiplication, then the fundamental properties of the FRDT have been presented in Table 1, where $\Gamma$ denotes the gamma function defined by $\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt, z \in \mathbb{C}$, is the continuous extension to the factorial function, and the function $\delta$ is defined by

$$
\delta(k) := \begin{cases} 
1 & \text{if } k = 0 \\
0 & \text{otherwise} \end{cases}.
$$

<table>
<thead>
<tr>
<th>Original Function $f(x, t)$</th>
<th>Fractional Reduced Differential Transformed Function $R_D{f(x, t)} = F_k(x)$</th>
</tr>
</thead>
</table>

Table 1: Basic properties of the FRDTM
\[ u(x,t)v(x,t) \quad U_k(x) \otimes V_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x) \]

\[ a_1u(x,t) + a_2v(x,t) \quad a_iU_k(x) \pm a_iV_k(x) \]

\[ x^m t^n u(x,t) \quad \begin{cases} x^m U_{k-n}(x), & \text{if } k \geq n \\ 0, & \text{else.} \end{cases} \]

\[ D_{t}^\alpha(u(x,t)) \quad \frac{\Gamma(1+(k+N)\alpha)}{\Gamma(1+k\alpha)} U_{k+N}(x) \]

\[ D_x^\alpha u(x,t) \quad D_x^\alpha U_k(x) \]

\[ e^{zt} \quad \frac{\lambda^k}{k!} \]

---

**Definition 3.2** The Mittag-Leffler function \( E_\alpha(z) \) with \( \alpha > 0 \) is defined by the following series representation, is valid in the whole complex plane

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)} \quad (14) \]

which is an advanced form of \( \exp(z) \), in particular, \( \exp(z) = \lim_{\alpha \to 1} E_\alpha(z) \).

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**4. Numerical Computations**

In this section, we describe the method explained in the [Section 2](#) using the following three examples of the time fractional Fokker-Planck equation (FPE) to validate the efficiency and reliability of the FRDTM.

**Example 4.1:** Consider the following time fractional order Fokker-Planck equation [29, 42]

\[ D_x^\alpha u = -D_x \left( xu \right) + D_x^2 \left( \frac{x^2 u}{2} \right), x,t > 0, 0 < \alpha \leq 1 \quad (15) \]

subject to the initial condition

\[ u(x,0) = x. \quad (16) \]

Applying the RDTM to [Eq. (15)] we obtain the following recurrence relation

\[ \frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) = -R_D \left[ D_x \left( xu \right) \right] + R_D \left[ D_x^2 \left( \frac{x^2 u}{2} \right) \right]. \quad (17) \]

Using the RDTM to the initial condition (14), we obtain the expression

\[ U_0(x) = x. \quad (18) \]
Using Eq. (18) into Eq. (17) we get the following $U_k(x)$ values successively

$$U_1(x) = \frac{x}{\Gamma(1+\alpha)}, U_2(x) = \frac{x}{\Gamma(1+2\alpha)}, \ldots, U_k(x) = \frac{x}{\Gamma(1+k\alpha)}, \ldots$$ (19)

Using the differential inverse reduced transform of $U_k(x)$, we have

$$u(x,t) = U_0(x) + \sum_{k=1}^{\infty} U_k(x) t^{k\alpha} = x \sum_{k=0}^{\infty} \left( \frac{1}{\Gamma(1+k\alpha)} \right) t^{k\alpha} = x \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(1+k\alpha)} = x E_\alpha(t^\alpha),$$ (20)

where $E_\alpha(t^\alpha)$ is the well known Mittag-Leffler function. It is noticed that solution (20) is in full agreement with the solutions obtained by HPM [29] and HPTM [42]. In particular, for $\alpha \to 1$ in Eq. (15), we get

$$u(x,t) = xe^t,$$ (21)

which is the exact solution for the Fokker-Planck Eq. (15) with $\alpha = 1$.

![Fig. 1. Comparison between FRDTM solution and exact solution of Example 4.1 for $\alpha = 1, t = 1$.](image)

![Fig. 2. Comparison between (a) exact solution and (b) approx. FRDTM solution of Example 4.1 for $\alpha = 1, t = 1$.](image)
Fig. 3. Physical behavior of approximate FRDTM solution $u(x,t)$ of Example 4.1 vs. $x$ at $t=1$ (left) and vs. $t$ at $x=1$ (right) for different values of $\alpha$.

Fig. 1 shows the comparison between the exact solution and approx. FRDTM solution while Fig. 2 shows the comparison between the surface (a) exact solution and (b) approx. FRDTM solution of Example 4.1. It is clearly seen from Fig. 1 and Fig. 2 that the solution obtained by FRDTM is identical with the exact solution for classical Foker-Plank Eq. (14).

Fig. 3 shows the physical behavior of the approximate FRDTM solution for different fraction Brownian motion $\alpha = 0.6, 0.7, 0.8, 0.9$ vs. $x$ at $t=1$ and vs. $t$ at $x=1$ respectively of Example 4.1.

**Example 4.2:** Consider the following time fractional order Fokker-Planck equation [29, 42]:

$$D_x^\alpha u = -D_x \left( \frac{\alpha}{6} \right) + D_x^2 \left( \frac{\alpha^2}{12} \right) , x, t > 0, 0 < \alpha \leq 1,$$

under the initial condition $u(x,0) = x^2$.

Applying the RDTM to Eq. (22), we obtain the following recurrence relation

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) = -R_0 \left[ D_x \left( \frac{\alpha}{6} \right) \right] + R_0 \left[ D_x^2 \left( \frac{\alpha^2}{12} \right) \right].$$

Using the RDTM in Eq. (23), we obtain the expression $U_0(x) = x^2$.

Using Eq. (25) into Eq. (24) we get the following $U_k(x)$ values successively

$$U_1(x) = \frac{x^2}{2\Gamma(1+\alpha)}, U_2(x) = \frac{x^2}{2^2\Gamma(1+2\alpha)}, \ldots, U_k(x) = \frac{x^2}{2^k\Gamma(1+k\alpha)}, \ldots$$. 
Using the differential inverse reduced transform of $U_k(x)$, we have

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = U_0(x) + \sum_{k=1}^{\infty} U_k(x) t^{k\alpha} = x^2 \sum_{k=0}^{\infty} \left( \frac{1}{\Gamma(1+k\alpha)} \right) \left( \frac{t^{\alpha}}{2} \right)^k$$

$$= x^2 \sum_{k=0}^{\infty} \frac{\left( \frac{t^{\alpha}}{2} \right)^k}{2^k \Gamma(1+k\alpha)} = x^2 E_{\alpha} \left( \frac{t^{\alpha}}{2} \right).$$

The same exact solution was obtained by HPM [29] and HPTM [42]. In particular, for $\alpha \to 1$ in Eq. (22), we get

$$u(x,t) = x^2 e^{\frac{t}{2}},$$

which is the exact solution for the classical Fokker-Planck Eq. (22) with $\alpha = 1$.

![Fig. 4. Comparison between exact solutions and FRDTM solutions of Example 4.2 for $\alpha = 1, t = 1$.](image)

![Fig. 5. Comparison between Exact solution (a) and approx. FRDTM solution (b) for $\alpha = 1, t = 1$, Example 4.2.](image)
Fig. 6 Physical characteristics of approximate FRDTM solution $u(x, t)$ of Example 4.2 vs. $x$ at $t = 1$ (left) and vs. $t$ at $x = 1$ (right) for different values of $\alpha$.

Fig. 4 depicts the comparison between the exact solution and approx. FRDTM solution for while Fig. 5 shows the comparison between the surface of (a) exact solution and (b) approx. FRDTM solution of Example 4.2. Fig. 4 and Fig. 5 show that FRDTM solution is identical with the exact solution of [21] when $\alpha = 1$. Fig. 6 depicts the physical characteristics of the approximate FRDTM solution for different fraction Brownian motion $\alpha = 0.6, 0.7, 0.8, 0.9$ vs. $x$ at $t = 1$ and vs. $t$ at $x = 1$, respectively of Example 4.2.

**Example 4.3:** Consider the following nonlinear time fractional Fokker-Planck equation [29, 42]

$$D_t^\alpha u = -D_x^2 \left( \frac{4u^2}{x} - \frac{xu}{3} \right) + D_x^2 \left( u^2 \right), x, t > 0, 0 < \alpha \leq 1$$

subject to the initial condition

$$u(x, 0) = x^2.$$ (30)

Applying the RDTM to Eq. (29) we obtain the following recurrence relation

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) = -R_p \left[ D_1^x \left( \frac{4u^2}{x} - \frac{xu}{3} \right) + R_p \left[ D_x^2 \left( u^2 \right) \right] \right].$$ (31)

Using the RDTM to the initial condition (30), we obtain the expression

$$U_0(x) = x^2.$$ (32)

Using Eq. (32) into Eq. (31), we get the following $U_k(x)$ values successively

$$U_1(x) = \frac{x^2}{\Gamma(1 + \alpha)}, U_2(x) = \frac{x^2}{\Gamma(1 + 2\alpha)}, \ldots, U_k(x) = \frac{x^2}{\Gamma(1 + k\alpha)}.$$ (33)

Using the differential inverse reduced transform of $U_k(x)$, we have
which is exactly the same as obtained in [29] using HPM, and in [42] using HPTM. In particular, for \( \alpha \to 1 \) in Eq. (29), we get
\[
u(x,t) = x^2 e^t\]
which is the exact solution for the classical Fokker-Planck Eq. (29) with \( \alpha = 1 \).

![Fig. 7](image_url)

Fig. 7. Comparison between exact solutions and FRDTM solutions of Example 4.3 for \( \alpha = 1, t = 1 \).

![Fig. 8](image_url)

Fig. 8. Comparison between Exact solution (a) and approx. FRDTM solution (b) for \( \alpha = 1, t = 1 \), Example 4.3.
Fig. 9. Physical signature of approximate FRDTM solution \( u(x, t) \) of Example 4.3 vs. \( x \) at \( t = 1 \) (left) and vs. \( t \) at \( x = 1 \) (right) for different values of \( \alpha \).

Fig. 7 shows the comparison between the exact solution and approx. FRDTM solution whereas Fig. 8 depicts the comparison between the surface of (a) exact solution and (b) approx. FRDTM solution of Example 4.3. It evident from Fig. 7 and Fig. 8 that FRDTM solution is identical with the exact solution for classical FPE \((28)\). Fig. 9 shows the physical signature of the approximate FRDTM solution for different fraction Brownian motion \( \alpha = 0.6, 0.7, 0.8, 0.9 \) vs. \( x \) at \( t = 1 \) and vs. \( t \) at \( x = 1 \) respectively of Example 4.3.

5. Conclusions

In this article, a new approximate solution of time-fractional FPE arising in several physical dynamical systems has been obtained by means of FRDTM with appropriate initial condition. The proposed approximate solutions of time-fractional FPE are obtained in terms of a power series, without using any kind of discretization, perturbation, or any other restrictive condition, etc. The validity and efficiency of FRDTM is illustrated by considering three numeric experiments. This method is very powerful analytical approach which can be easily applicable to broad classes of real world problems arising in physical dynamical systems and engineering.

References